NDA-FP-23
January (1996)

Group Theoretical Structure of $N = 1$ and $N = 2$ Two-Form Supegravity

Shin'ichi NOJIRI

Department of Mathematics and Physics
National Defence Academy
Hashirimizu, Yokosuka 139, JAPAN

ABSTRACT

We clarifies the group theoretical structure of $N = 1$ and $N = 2$ two-form supergravity, which is classically equivalent to the Einstein supergravity. $N = 1$ and $N = 2$ two-form supergravity theories can be formulated as gauge theories. By introducing two Grassmann variables θ^A $(A = 1, 2)$, we construct the explicit representations of the generators Q^i of the gauge group, which makes to express any product of the generators as a linear combination of the generators $Q^iQ^j = \sum_k f^{ij}_k Q^k$. By using the expression and the tensor product representation, we explain how to construct finite-dimensional representations of the gauge groups. Based on these representations, we construct the Lagrangeans of $N = 1$ and $N = 2$ two-form supergravity theories.

1 Introduction

The Einstein gravity theory might be an effective theory of a more fundamental theory *e.g.* superstring theory since the action of the Einstein gravity is not renormalizable. Two-form gravity theory is known to be classically equivalent to the Einstein gravity theory and is obtained from a topological field theory, which is called BF theory [1], by imposing constraint conditions [2]. The BF theory has a large local symmety called the Kalb-Ramond symmetry [3]. Since the Kalb-Ramond symmetry is very stringy symmetry, the fundamental gravity theory is expected to be a kind of string theory [4]. The extension of the two-form gravity theory to the supergravity theory was considered in Ref.[5]. Furthermore the supergravity theory which has a cosmological term or $N = 2$ supersymmetry was proposed in Ref.[6] and the group theoretical structure of these supergravity theory was discussed in Ref.[7]. $N = 1$ and $N = 2$ two-form supergravity theories can be formulated as gauge theories. In this paper, we call the gauge algebras as $N = 1$ and $N = 2$ topological superalgebras (TSA), which are the subalgebras of $N = 1$ and $N = 2$ Neveu-Schwarz algebaras whose generators are $(L_0, L_{\pm1}, G_{\pm\frac{1}{2}})$ and $(L_0, L_{\pm1}, G^{\pm}_{\pm\frac{1}{2}}, J_0)$, respectively. $N = 1$ topological superalgebra is nothing but $osp(1, 2)$ algebra.

In this paper, by introducing two Grassmann variables θ^A $(A = 1, 2)$, we construct the explicit representations of the generators Q^i, which makes to express any product of the generators as a linear combination of the generators $Q^iQ^j = \sum_k f^{ij}_k Q^k$. By using the expression and the direct product representation, we explain how to construct finite-dimensional representations of the gauge groups. The representation theory makes it possible to construct a general action of two-form $N = 1$ and $N = 2$ supergravity theories, which is expected to give a clue for the non-perturbative analysis of the supergravity. The non-perturbative analysis is also partially given in this paper.

This paper is organized as follows: In section 2, we give the representation theories of $N = 1$ and $N = 2$ topological superalgebra. By using the representation, we construct the Lagrangeans of $N = 1$ and $N = 2$ two-form supergravities in section 3. In section 4, we investigate the symmetry of the system and consider the non-perturbative effect. The last section is devoted to summary.

2 The Representaions of $N = 1$ and $N = 2$ Topological Superalgebras

By using two Grassmann (anti-commuting) variables θ^A ($A = 1, 2$), we define the following generators,

$$
\begin{aligned}
G_A &= \frac{\partial}{\partial\theta^A}, \quad T^a = \theta^A T^{a\ B}_{\ A} \frac{\partial}{\partial\theta^B} \\
I &= \theta^A \frac{\partial}{\partial\theta^A}, \quad H^A = \theta^A\theta^B \frac{\partial}{\partial\theta^B}
\end{aligned}
\tag{1}
$$

Here

$$
T^{a\ B}_{\ A} = \frac{1}{2}\sigma^{a\ B}_{\ A} \tag{2}
$$

and σ^a's ($a = 1, 2, 3$) are Pauli matrices. These generators make the following algebra, which we call topological superconformal algebra (TSCA) in this paper:

$$
\begin{aligned}
&\{G_A, H^B\} = \frac{1}{2}\delta_A^{\ B} I - 2T^{a\ B}_{\ A} T^a \\
&[G_A, T^a] = T^{a\ B}_{\ A} G_B, \quad [T^a, H^A] = T^{a\ A}_{\ B} H^B \\
&[T^a, T^b] = i\epsilon^{abc}T^c, \quad \{G_A, G_B\} = \{H^A, H^B\} = 0
\end{aligned}
\tag{3}
$$

This algebra contains a closed subalgebra, which is found by defining an operator $\hat{G}_A$:

$$
\hat{G}_A \equiv G_A + \alpha\epsilon_{AB}H^B \tag{4}
$$

Here α is a parameter which can be absorbed into the redefinition of the operators but we keep α as a free parameter for later convenience. Then $\hat{G}_A$ and T^a make a closed algebra:

$$
\begin{aligned}
&\{\hat{G}_A, \hat{G}_B\} = -4\alpha T^a_{AB} T^a \\
&[\hat{G}_A, T^a] = T^{a\ B}_{\ A} \hat{G}_B, \quad [T^a, T^b] = i\epsilon^{abc}T^c
\end{aligned}
\tag{5}
$$

Here T^a_{AB} is defined by

$$
T^a_{AB} \equiv \epsilon_{BC} T^{a\ C}_{\ A} \tag{6}
$$

and

$$
\begin{aligned}
\epsilon^{AB} &= -\epsilon^{BA} \\
\epsilon_{AB} &= -\epsilon_{BA} \\
\epsilon^{12} &= \epsilon_{21} = 1\ .
\end{aligned}
\tag{7}
$$

The algebra (5) is nothing but $osp(1,2)$ algebra which is the subalgebra of the Neveu-Schwarz algebra whose generators are L_0, $L_{\pm 1}$ and $G_{\pm\frac{1}{2}}$. In this paper, we call this algebra (5) as $N=1$ topological superalgebra (TSA). As we will see later, $\hat{G}_A$ generates left-handed supersymmetry.

By defining the following operators,

$$\begin{aligned}
J &= -\frac{8}{3}\alpha T^a T^a + \epsilon^{AB}\hat{G}_A\hat{G}_B \\
G^1_A &= \hat{G}_A \\
G^2_A &= \frac{4}{3} i T^a{}_A{}^B (T^a\hat{G}_B + \hat{G}_B T^a) \ ,
\end{aligned} \tag{8}$$

we can also construct an algebra, which we call $N=2$ topological superalgebra $(k,l=1,2)$

$$\begin{aligned}
&\{G^k_A, \hat{G}^l_B\} = -4\delta^{kl}\alpha T^a_{AB}T^a + i\epsilon^{kl}\epsilon_{AB}J \\
&[G^k_A, T^a] = T^a{}_A{}^B G^k_B, \quad [T^a, T^b] = i\epsilon^{abc}T^c \\
&[J, G^i_A] = i\alpha\epsilon^{ij}G^j_A, \quad [T^a, J] = 0
\end{aligned} \tag{9}$$

This algebra is the subalgebra of $N=2$ Neveu-Schwarz algebra whose generators are L_0, $L_{\pm 1}$, $G^{(\pm)}_{\pm\frac{1}{2}}$ and J_0.

Since all the operators are explicitly given in terms of θ^A and $\frac{\partial}{\partial\theta^A}$, we can find that the product of operators is given by a linear combination of the operators:

$$\begin{aligned}
&G^k G^l = \delta^{kl}\left\{-2\alpha T^a_{AB}T^a - \epsilon_{AB}\left(\frac{3}{2}J + 2\alpha P\right)\right\} \\
&\qquad\quad + i\epsilon^{kl}\left(-2\alpha T^a_{AB}T^a + \frac{1}{2}\epsilon_{AB}J\right) \\
&T^a T^b = \frac{i}{2}\epsilon^{abc}T^c + \delta^{ab}\left(\frac{1}{4\alpha}J + \frac{1}{2}P\right) \\
&J^2 = 3\alpha J + 2\alpha^2 P \\
&G^k_A T^a = T^a G^k_A + T^a_A{}^B G^k_B = \frac{1}{2}T^a_A{}^B G^k_B - \frac{i}{2}\epsilon^{kl}T^a_A{}^B G^l_B \\
&J G^k_A = G^k_A J + i\alpha\epsilon^{kl}G^l_A = -\frac{1}{2}\alpha G^k_A + \frac{3}{2}i\alpha\epsilon^{kl}G^l_A \\
&J T^a = T^a J = -\alpha T^a
\end{aligned} \tag{10}$$

Here P is a projection operator

$$P^2 = P \tag{11}$$

defined by

$$P \equiv -\frac{1}{2\alpha}\epsilon^{AB}\hat{G}_A\hat{G}_B + 2T^aT^a \tag{12}$$

and P acts as unity on the operators

$$\begin{aligned} PG_A^k &= G_A^kP = G_A^k \\ PT^a &= T^aP = T^a \\ PJ &= JP = J \end{aligned} \tag{13}$$

Therefore the invariant trace of the product of the operators can be defined by the coefficients of P in Equation (10):

$$\begin{aligned} &\mathrm{tr}G^kG^l = -2\alpha\delta^{kl}\epsilon_{AB}, \quad \mathrm{tr}T^aT^b = \frac{1}{2}\delta^{ab}, \quad \mathrm{tr}J^2 = 2\alpha^2 \\ &\mathrm{tr}G_A^kT^a = \mathrm{tr}T^aG_A^k = \mathrm{tr}JG_A^k = \mathrm{tr}G_A^k\mathrm{tr}JT^a = \mathrm{tr}T^aJ = 0 \end{aligned} \tag{14}$$

We express the product law (10) of the operators Q^i ($Q^i = T^a$, G_A^k, J and P) by

$$Q^iQ^j = \sum_k f_k^{ij}Q^k \ . \tag{15}$$

Especially the expression of the invariant trace (14) is given by

$$\mathrm{tr}Q^iQ^j = g^{ij} \equiv f_P^{ij} \tag{16}$$

The representation of $N = 1$ superalgebra is given by a doublet of the representations $\left(p, p + \frac{1}{2}\right)$ in $SU(2)$ (p is an integer or half-integer), which is generated by T^a and that of $N = 2$ is given by a quartet $\left(p, p + \frac{1}{2}, p + \frac{1}{2}, p + 1\right)$.

$\left(\frac{1}{2}, 1\right)$ representation of $N = 1$ superalgebra is giben by $(\hat{G}_A, T^a)$ and $\left(0, \frac{1}{2}\right)$ is given by (J, G_A^2). (J, G_A^k, T^a) makes the $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$ representation of $N = 2$ superalgebra.

$\left(1, \frac{3}{2}\right)$ and $\left(\frac{3}{2}, 2\right)$ representations of $N = 1$ superalgebra are given by a tensor product, where $\hat{G}_A$ and T^a are replaced by $\hat{G}_A \otimes P + P \otimes \hat{G}_A$ and $T^a \otimes P + P \otimes T^a$:

- $\left(1, \frac{3}{2}\right)$ reprezentation (K_{AB}, N^2_{ABC}):

$$
\begin{aligned}
K_{AB} &= -T^a_{AB}(J \otimes T^a + T^a \otimes J) - i\frac{1}{2}\epsilon_{kl} G^k_{(A} \otimes G^l_{B)} \\
N^2_{ABC} &= T^a_{(AB}(G^2_{C)} \otimes T^a + T^a \otimes G^2_{C)})
\end{aligned} \tag{17}
$$

  Here $(AB \cdots X)$ means a symmetrization with respect to the indeces $AB \cdots X$.

- $\left(\frac{3}{2}, 2\right)$ representation (N^1_{ABC}, M_{ABCD}):

$$
\begin{aligned}
N^1_{ABC} &= T^a_{(AB}(G^1_{C)} \otimes T^a + T^a \otimes G^1_{C)}) \\
M_{ABCD} &= T^a_{(AB} T^b_{CD)} T^a \otimes T^b
\end{aligned} \tag{18}
$$

$(K_{AB}, N^k_{ABC}, M_{ABCD})$ makes $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$ representation of $N = 2$ superalgebra. The commutater of G^k_A with $(K_{AB}, N^k_{ABC}, M_{ABCD})$ are given by[1]

$$
\begin{aligned}
&[G^k_E, M_{ABCD}] = -\frac{1}{2}\epsilon_{E(A} N^k_{BCD)} \\
&\{G^k_D, N^l_{ABC}\} = -8\alpha\delta^{kl} M_{ABCD} - i\epsilon^{kl}\epsilon_{D(A} K_{BC)} \\
&[G^k_C, K_{AB}] = -3i\epsilon^{kl} N^l_{ABC} \ .
\end{aligned} \tag{19}
$$

The coefficient of $P \otimes P$ in the product of K_{AB}, N^k_{ABC} and M_{ABCD} gives the invariant trace

$$
\begin{aligned}
\mathrm{tr} M_{ABCD} M_{A'B'C'D'} &= \frac{1}{2}\epsilon_{A(A'}\epsilon_{\hat{B}B'}\epsilon_{\hat{C}C'}\epsilon_{\hat{D}D')} \\
\mathrm{tr} N^k_{ABC} N^l_{A'B'C'} &= \alpha\epsilon_{A(A'}\epsilon_{\hat{B}B'}\epsilon_{\hat{C}C')} \\
\mathrm{tr} K_{AB} K_{A'B'} &= \alpha^2\epsilon_{A(A'}\epsilon_{\hat{B}B')}
\end{aligned} \tag{20}
$$

Here $(AB \cdots \hat{F} \cdots Z)$ means the symmetrization with respect to the indeces $AB \cdots Z$ *except* F.

[1] G^k_A in Equation (19) is understood to be $G^k_A \otimes P + P \otimes G^k_A$.

3 The Lagrangeans of $N = 1$ and $N = 2$ Two-Form Supergravities

In order to construct the Lagrangean of $N = 1$ two-form supergravity theory, we introduce the gauge field A_μ which is $\left(\frac{1}{2}, 1\right)$ representation

$$A_\mu = \psi_\mu^A \hat{G}_A + \omega_\mu^a T^a \tag{21}$$

and define the field strength as follows

$$\begin{aligned} R_{\mu\nu} &= [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \\ &= \{\partial_\mu \psi_\nu^A - \partial_\nu \psi_\mu^A + T^a_B{}^A (\psi_\mu^B \omega_\nu^a - \psi_\nu^B \omega_\mu^a)\} \hat{G}_A \\ &\quad + \{\partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a + i\epsilon^{abc} \omega_\mu^b \omega_\nu^c + 4\alpha T^a_{AB} \psi_\mu^A \psi_\nu^B\} T^a \end{aligned} \tag{22}$$

The left-handed supersymmetry transformation law of the gauge fields is given by

$$\begin{aligned} \delta_G A_\mu &= [\epsilon^A \hat{G}_A, \partial_\mu + A_\mu] \\ &= (-\partial_\mu \epsilon^A + T^a_B{}^A \epsilon^B \omega_\mu^a) \hat{G}_A + 4\alpha T^a_{AB} \epsilon^A \psi_\mu^B T^a \end{aligned} \tag{23}$$

We also introduce the two-form field $X_{\mu\nu}$ which is $\left(\frac{1}{2}, 1\right)$ representation:

$$X_{\mu\nu} = \chi_{\mu\nu}^A \hat{G}_A + \Sigma_{\mu\nu}^a T^a \tag{24}$$

Then the Lagrangean $\mathcal{L}_{\rm BF}$ of the so-called BF theory with $N = 1$ local supersymmetry is given by

$$\mathcal{L}_{\rm BF} = \epsilon^{\mu\nu\rho\sigma} \left\{ \frac{1}{g} {\rm tr} R_{\mu\nu} X_{\rho\sigma} + \Lambda {\rm tr} X_{\mu\nu} X_{\rho\sigma} \right\} \tag{25}$$

$$\begin{aligned} {\rm tr} R_{\mu\nu} X_{\rho\sigma} &= 2\alpha \epsilon_{AB} \{\partial_\mu \psi_\nu^A - \partial_\nu \psi_\mu^A + T^a_B{}^A (\psi_\mu^B \omega_\nu^a - \psi_\nu^B \omega_\mu^a)\} \chi_{\rho\sigma}^B \\ &\quad + \frac{1}{2} \{\partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a + i\epsilon^{abc} \omega_\mu^b \omega_\nu^c + 4\alpha T^a_{AB} \psi_\mu^A \psi_\nu^B\} \Sigma_{\rho\sigma}^a \end{aligned} \tag{26}$$

$${\rm tr} X_{\mu\nu} X_{\rho\sigma} = 2\alpha \epsilon_{AB} \chi_{\mu\nu}^A \chi_{\rho\sigma}^B + \frac{1}{2} \Sigma_{\mu\nu}^a \Sigma_{\rho\sigma}^a \tag{27}$$

Here g is a gauge coupling constant and Λ is a cosmological constant. In order to obtain $N = 1$ two-form supergravity theory, we need to introduce the multiplier field Φ which is $\left(\frac{3}{2}, 2\right)$ representation:

$$\Phi = \kappa^{ABC} N_{ABC} + \phi^{ABCD} M_{ABCD} \tag{28}$$

The Lagrangean $\mathcal{L}$ of $N=1$ two-form supergravity is given by adding the constraint term to the Lagrangean $\mathcal{L}_{\rm BF}$:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\rm BF} + \epsilon^{\mu\nu\rho\sigma}{\rm tr}\Phi(X_{\mu\nu}\otimes X_{\rho\sigma}) \\
{\rm tr}\Phi(X_{\mu\nu}\otimes X_{\rho\sigma}) &= -\alpha T^a_{AB}\epsilon_{CD}\kappa^{ABC}(\chi^D_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\chi^D_{\rho\sigma}) \\
&\quad + T^a_{AB}T^b_{CD}\phi^{ABCD}\Sigma^a_{\mu\nu}\Sigma^b_{\rho\sigma}
\end{aligned} \tag{29}
$$

The Lagrangean of $N=2$ theory is also given by introducing gauge field which is $\left(0,\frac{1}{2},\frac{1}{2},1\right)$ representation

$$
A_\mu = B_\mu J + \psi^{kA}_\mu G^k_A + \omega^a_\mu T^a \tag{30}
$$

and defining the field strength

$$
\begin{aligned}
R_{\mu\nu} &= [\partial_\mu + A_\mu, \partial_\nu + A_\nu] \\
&= \{\partial_\mu B_\nu - \partial_\nu B_\mu - i\epsilon_{AB}\epsilon^{kl}\psi^{k\ A}_\mu \psi^{l\ B}_\nu\}J \\
&\quad + \{\partial_\mu\psi^{k\ A}_\nu - \partial_\nu\psi^{k\ A}_\mu + T^a_B{}^A(\psi^{k\ B}_\mu\omega^a_\nu - \psi^{k\ B}_\nu\omega^a_\mu) \\
&\quad - i\epsilon^{kl}(B_\mu\psi^{l\ A}_\nu - B_\nu\psi^{l\ A}_\nu)\}\hat{G}_A \\
&\quad + \{\partial_\mu\omega^a_\nu - \partial_\nu\omega^a_\mu + i\epsilon^{abc}\omega^b_\mu\omega^c_\nu + 4\alpha T^a_{AB}\psi^{k\ A}_\mu\psi^{k\ B}_\nu\}T^a
\end{aligned} \tag{31}
$$

The gauge transformation law of the gauge field has the following form:

$$
\begin{aligned}
\delta A_\mu &= [aJ + \epsilon^{kA}G^k_A + \delta^a T^a, \partial_\mu + A_\mu] \\
&= (-\partial_\mu a - i\epsilon_{AB}\epsilon^{kl}\epsilon^{kA}\psi^{lB}_\mu)J \\
&\quad + \left\{-\partial_\mu\epsilon^{kA} + T^a_B{}^A(\epsilon^{kB}\omega^a_\mu + i\delta^a\psi^{kB}_\mu) - i\alpha\epsilon^{kl}(a\psi^{lA}_\mu - \epsilon^{lA}B_\mu)\right\}G^k_A \\
&\quad + (-\partial_\mu\delta^a + i\epsilon^{abc}\delta^b\omega^c_\mu + 4\alpha T^a_{AB}\epsilon^{kA}\psi^{kB}_\mu)T^a
\end{aligned} \tag{32}
$$

The two-form field in $N=2$ theory is $\left(0,\frac{1}{2},\frac{1}{2},1\right)$ representation

$$
X_{\mu\nu} = \Pi_{\mu\nu}J + \chi^{kA}_{\mu\nu}G^k_A + \Sigma^a_{\mu\nu}T^a \tag{33}
$$

Then the Lagrangean of $N=2$ BF theory is given by

$$
\begin{aligned}
\mathcal{L}_{\rm BF} &= \epsilon^{\mu\nu\rho\sigma}\left\{\frac{1}{g}{\rm tr}R_{\mu\nu}X_{\rho\sigma} + \Lambda{\rm tr}X_{\mu\nu}X_{\rho\sigma}\right\} \qquad (34) \\
{\rm tr}R_{\mu\nu}X_{\rho\sigma} &= 2\alpha^2\{\partial_\mu B_\nu - \partial_\nu B_\mu - i\epsilon_{AB}\epsilon^{kl}\psi^{k\ A}_\mu\psi^{l\ B}_\nu\}\Pi_{\rho\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&+2\alpha\{\partial_\mu \psi_\nu^{k\ A} - \partial_\nu \psi_\mu^{k\ A} + T_B^{a\ A}(\psi_\mu^{k\ B}\omega_\nu^a - \psi_\nu^{k\ B}\omega_\mu^a) \\
&-i\epsilon^{kl}(B_\mu \psi_\nu^{l\ A} - B_\nu \psi_\nu^{l\ A})\}\chi_{\rho\sigma}^{kB} \\
&+\frac{1}{2}\{\partial_\mu \omega_\nu^a - \partial_\nu \omega_\mu^a + i\epsilon^{abc}\omega_\mu^b \omega_\nu^c + 4\alpha T_{AB}^a \psi_\mu^{k\ A}\psi_\nu^{k\ B}\}\Sigma_{\rho\sigma}^a \qquad (35)
\end{aligned}
$$

$$
\mathrm{tr} X_{\mu\nu} X_{\rho\sigma} = 2\alpha^2 \Pi_{\mu\nu}\Pi_{\rho\sigma} + 2\alpha\epsilon_{AB}\chi_{\mu\nu}^{kA}\chi_{\rho\sigma}^{kB} + \frac{1}{2}\Sigma_{\mu\nu}^a \Sigma_{\rho\sigma}^a \qquad (36)
$$

The Lagrangean $\mathcal{L}$ of $N=2$ two-form supergravity theory is given by introducing the multiplier field which is $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$ representation

$$
\Phi = \lambda^{AB} K_{AB} + \kappa^{kABC} N_{ABC}^k + \phi^{ABCD} M_{ABCD} \qquad (37)
$$

and adding the term which gives the constraint on the two-form field

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{BF}} + \epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma}) \qquad (38) \\
\mathrm{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma}) &= \alpha^2 \lambda^{AB}\{-T_{AB}^a(\Pi_{\mu\nu}\Sigma_{\rho\sigma}^a + \Sigma_{\mu\nu}^a \Pi_{\rho\sigma}) + i\epsilon^{kl}\epsilon_{AC}\epsilon_{BD}\chi_{\mu\nu}^{kC}\chi_{\rho\sigma}^{lD}\} \\
&\quad -\alpha T_{AB}^a \epsilon_{CD}\kappa^{kABC}(\chi_{\mu\nu}^{kD}\Sigma_{\rho\sigma}^a + \Sigma_{\mu\nu}^a \chi_{\rho\sigma}^{kD}) \\
&\quad +T_{AB}^a T_{CD}^b \phi^{ABCD}\Sigma_{\mu\nu}^a \Sigma_{\rho\sigma}^b \qquad (39)
\end{aligned}
$$

4 The Symmetry of the Lagrangeans

We now consider the right-handed supersymmetry. The Lagrangeans of the $N=1$ and $N=2$ have the following form

$$
\mathcal{L} = \epsilon^{\mu\nu\rho\sigma}\left\{\frac{1}{g}\mathrm{tr} R_{\mu\nu} X_{\rho\sigma} + \Lambda \mathrm{tr} X_{\mu\nu} X_{\rho\sigma} + \mathrm{tr}\Phi(X_{\mu\nu} \otimes X_{\rho\sigma})\right\} \qquad (40)
$$

On the other hand the Lagrangeans of the corresponding BF theory have the following form

$$
\mathcal{L}_{\mathrm{BF}} = \epsilon^{\mu\nu\rho\sigma}\left\{\frac{1}{g}\mathrm{tr} R_{\mu\nu} X_{\rho\sigma} + \Lambda \mathrm{tr} X_{\mu\nu} X_{\rho\sigma}\right\} \qquad (41)
$$

The Lagrangean (41) has the large local symmetry which is called Kalb-Ramond symmetry. The parameter of the transformation C_μ is $\left(\frac{1}{2}, 1\right)$ representation in $N=1$ theory and $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$ representation in $N=2$ theory

and the transformation law of the Kalb-Ramond symmetry is given by

$$\begin{aligned}\delta_{\rm KR} A_\mu &= -g\Lambda C_\mu \\ \delta_{\rm KR} X_{\mu\nu} &= \frac{1}{2}(D_\mu C_\nu - D_\nu C_\mu)\end{aligned} \tag{42}$$

Here the covariant derivative D_μ is defined by

$$D_\mu \,\cdot\, = [\partial_\mu + A_\mu,\ \cdot\ ] \tag{43}$$

Now we consider the Kalb-Ramond like transformation for the Lagrangean (40):

$$\begin{aligned}\delta_{\rm KR} A_\mu &= -g\Lambda C_\mu - g\Phi\times C_\mu \\ \delta_{\rm KR} X_{\mu\nu} &= \frac{1}{2}(D_\mu C_\nu - D_\nu C_\mu)\end{aligned} \tag{44}$$

Here the product $R\times S$ of two operators $R=\sum_{ij} r_{ij}Q^i\otimes Q^j$, which is $\left(\frac{3}{2},2\right)$ representation in $N=1$ theory and $\left(1,\frac{3}{2},\frac{3}{2},2\right)$ representation in $N=2$ theory, and $S=\sum_i s_iQ^i$, which is $\left(\frac{1}{2},1\right)$ representation in $N=1$ theory and $\left(0,\frac{1}{2},\frac{1}{2},1\right)$ representation in $N=2$ theory, is defined by

$$R\times S\equiv\sum_{ijk} s_i r_{jk} g^{ik} G^j \tag{45}$$

Here g^{ik} is defined in Equation (16). The product $R\times S$ is $\left(\frac{3}{2},2\right)$ representation in $N=1$ theory and $\left(1,\frac{3}{2},\frac{3}{2},2\right)$ representation in $N=2$ theory. Then the change of the Lagrangean (40) is given by

$$\delta_{\rm KR}\mathcal{L} = -\epsilon^{\mu\nu\rho\sigma}{\rm tr} D_\mu\Phi C_\nu\Sigma_{\rho\sigma} + \text{ total derivative} \tag{46}$$

This tells that the Lagrangean (40) is invariant if the parameter C_μ satisfies the equation

$$0=\epsilon^{\mu\nu\rho\sigma}B_\nu\otimes\Sigma_{\rho\sigma}\big|_{\left(\frac{3}{2},2\right)\text{ or }\left(1,\frac{3}{2},\frac{3}{2},2\right)\text{ part}}\ . \tag{47}$$

Equation (47) has non-trivial solutions and the fermionic part of the solution corresponds to right-handed supersymmetry [6]. The commutator of the right-handed supersymmetry transformation and the left-handed one contains the general coordinate transformation.

When $\alpha \neq 0$, the parameter α can be absorbed into the redefinition of the operators or fields as follows:

$$
\begin{aligned}
&\omega_\mu^a \to \omega_\mu^a, \quad \psi_\mu^{kA} \to \alpha^{-\frac{1}{2}}\psi_\mu^{kA}, \quad B_\mu \to \alpha^{-1}B_\mu \\
&\Sigma_{\mu\nu}^a \to \Sigma_{\mu\nu}^a, \quad \chi_{\mu\nu}^{kA} \to \alpha^{-\frac{1}{2}}\chi_{\mu\nu}^{kA}, \quad \Pi_{\mu\nu} \to \alpha^{-1}\Pi_{\mu\nu} \\
&\phi^{ABCD} \to \phi^{ABCD}, \quad \kappa^{ABC} \to \alpha^{-\frac{1}{2}}\kappa^{ABC}, \quad \lambda^{AB} \to \alpha^{-1}\lambda^{AB} \ .
\end{aligned} \tag{48}
$$

Then $N=1$ Lagrangean has the following form

$$
\begin{aligned}
\mathcal{L} \ = \ & \epsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{g}\Big\{2\epsilon_{AB}\{\partial_\mu\psi_\nu^A - \partial_\nu\psi_\mu^A + T^a_B{}^A(\psi_\mu^B\omega_\nu^a - \psi_\nu^B\omega_\mu^a)\}\chi_{\rho\sigma}^B \\
& + \frac{1}{2}(\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a + i\epsilon^{abc}\omega_\mu^b\omega_\nu^c + 4T_{AB}^a\psi_\mu^A\psi_\nu^B)\Sigma_{\rho\sigma}^a\Big\} \\
& + \Lambda\left\{2\epsilon_{AB}\chi_{\mu\nu}^A\chi_{\rho\sigma}^B + \frac{1}{2}\Sigma_{\mu\nu}^a\Sigma_{\rho\sigma}^a\right\} \\
& - T_{AB}^a\epsilon_{CD}\kappa^{ABC}(\chi_{\mu\nu}^D\Sigma_{\rho\sigma}^a + \Sigma_{\mu\nu}^a\chi_{\rho\sigma}^D) \\
& + T_{AB}^aT_{CD}^b\phi^{ABCD}\Sigma_{\mu\nu}^a\Sigma_{\rho\sigma}^b\Big]
\end{aligned} \tag{49}
$$

and $N=2$ Lagrangean the following form

$$
\begin{aligned}
\mathcal{L} \ = \ & \epsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{g}\Big\{2\{\partial_\mu B_\nu - \partial_\nu B_\mu - i\epsilon_{AB}\epsilon^{kl}\psi_\mu^{k\ A}\psi_\nu^{l\ B}\}\Pi_{\rho\sigma} \\
& + 2\{\partial_\mu\psi_\nu^{k\ A} - \partial_\nu\psi_\mu^{k\ A} + T^a_B{}^A(\psi_\mu^{k\ B}\omega_\nu^a - \psi_\nu^{k\ B}\omega_\mu^a) \\
& - i\epsilon^{kl}(B_\mu\psi_\nu^{l\ A} - B_\nu\psi_\nu^{l\ A})\}\chi_{\rho\sigma}^{kB} \\
& + \frac{1}{2}(\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a + i\epsilon^{abc}\omega_\mu^b\omega_\nu^c + 4T_{AB}^a\psi_\mu^{k\ A}\psi_\nu^{k\ B})\Sigma_{\rho\sigma}^a\Big\} \\
& + \Lambda\Big\{2\Pi_{\mu\nu}\Pi_{\rho\sigma} + 2\epsilon_{AB}\chi_{\mu\nu}^{kA}\chi_{\rho\sigma}^{kB} + \frac{1}{2}\Sigma_{\mu\nu}^a\Sigma_{\rho\sigma}^a\Big\} \\
& + \lambda^{AB}\{-T_{AB}^a(\Pi_{\mu\nu}\Sigma_{\rho\sigma}^a + \Sigma_{\mu\nu}^a\Pi_{\rho\sigma}) + i\epsilon^{kl}\epsilon_{AC}\epsilon_{BD}\chi_{\mu\nu}^{kC}\chi_{\rho\sigma}^{lD}\} \\
& - T_{AB}^a\epsilon_{CD}\kappa^{kABC}(\chi_{\mu\nu}^{kD}\Sigma_{\rho\sigma}^a + \Sigma_{\mu\nu}^a\chi_{\rho\sigma}^{kD}) \\
& + T_{AB}^aT_{CD}^b\phi^{ABCD}\Sigma_{\mu\nu}^a\Sigma_{\rho\sigma}^b\Big]
\end{aligned} \tag{50}
$$

The Lagrangeans (49) and (50) are nothing but the Lagrangeans found in Ref.[6]. These Lagrangeans are invariant under the following $U(1)$ "symmetry"

$$
\omega_\mu^a \to \omega_\mu^a, \quad \psi_\mu^{kA} \to \psi_\mu^{kA}, \quad B_\mu \to B_\mu
$$

$$
\begin{array}{l}
\Sigma^a_{\mu\nu} \to \mathrm{e}^{\varphi}\Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \to \mathrm{e}^{\varphi}\chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \to \mathrm{e}^{\varphi}\Pi_{\mu\nu} \\
\phi^{ABCD} \to \mathrm{e}^{-2\varphi}\phi^{ABCD}, \quad \kappa^{ABC} \to \mathrm{e}^{-2\varphi}\kappa^{ABC}, \quad \lambda^{AB} \to \mathrm{e}^{-2\varphi}\lambda^{AB} \\
, g \to \mathrm{e}^{\varphi} g, \quad \Lambda \to \mathrm{e}^{-2\varphi}\Lambda
\end{array}
\tag{51}
$$

We can also consider $\alpha \to 0$ theory by redefining the fields as follows

$$
\begin{array}{l}
\omega^a_\mu \to \omega^a_\mu, \quad \psi^{kA}_\mu \to \psi^{kA}_\mu, \quad B_\mu \to B_\mu \\
\Sigma^a_{\mu\nu} \to \Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \to \alpha^{-1}\chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \to \alpha^{-2}\Pi_{\mu\nu} \\
\phi^{ABCD} \to \phi^{ABCD}, \quad \kappa^{ABC} \to \kappa^{ABC}, \quad \lambda^{AB} \to \lambda^{AB} \\
g \to g, \quad \Lambda \to \left\{ \begin{array}{ll} \alpha\Lambda & (N=1) \\ \alpha^2\Lambda & (N=2) \end{array} \right.
\end{array}
\tag{52}
$$

then $N=1$ Lagrangean is rewritten by

$$
\begin{aligned}
\mathcal{L} \;=\; & \epsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{g}\Big\{2\epsilon_{AB}\{\partial_\mu\psi^A_\nu - \partial_\nu\psi^A_\mu + T^a_B{}^A(\psi^B_\mu\omega^a_\nu - \psi^B_\nu\omega^a_\mu)\}\chi^B_{\rho\sigma} \\
& +\frac{1}{2}(\partial_\mu\omega^a_\nu - \partial_\nu\omega^a_\mu + i\epsilon^{abc}\omega^b_\mu\omega^c_\nu)\Sigma^a_{\rho\sigma}\Big\} \\
& +2\Lambda\epsilon_{AB}\chi^A_{\mu\nu}\chi^B_{\rho\sigma} \\
& -T^a_{AB}\epsilon_{CD}\kappa^{ABC}(\chi^D_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\chi^D_{\rho\sigma}) \\
& +T^a_{AB}T^b_{CD}\phi^{ABCD}\Sigma^a_{\mu\nu}\Sigma^b_{\rho\sigma}
\end{aligned}
\tag{53}
$$

The above Lagrangean with $\Lambda = 0$ was found in Ref.[5]. On the other hand the $N=2$ Lagrangean has the following form:

$$
\begin{aligned}
\mathcal{L} \;=\; & \epsilon^{\mu\nu\rho\sigma}\Big[\frac{1}{g}\Big\{2\{\partial_\mu B_\nu - \partial_\nu B_\mu - i\epsilon_{AB}\epsilon^{kl}\psi^{k\ A}_\mu\psi^{l\ B}_\nu\}\Pi_{\rho\sigma} \\
& +2\{\partial_\mu\psi^{k\ A}_\nu - \partial_\nu\psi^{k\ A}_\mu + T^a_B{}^A(\psi^{k\ B}_\mu\omega^a_\nu - \psi^{k\ B}_\nu\omega^a_\mu)\}\chi^{kB}_{\rho\sigma} \\
& +\frac{1}{2}(\partial_\mu\omega^a_\nu - \partial_\nu\omega^a_\mu + i\epsilon^{abc}\omega^b_\mu\omega^c_\nu)\Sigma^a_{\rho\sigma} \qquad (54) \\
& +2\Lambda\Pi_{\mu\nu}\Pi_{\rho\sigma} \\
& +\lambda^{AB}\{-T^a_{AB}(\Pi_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\Pi_{\rho\sigma}) + i\epsilon^{kl}\epsilon_{AC}\epsilon_{BD}\chi^{kC}_{\mu\nu}\chi^{lD}_{\rho\sigma}\} \\
& -T^a_{AB}\epsilon_{CD}\kappa^{kABC}(\chi^{kD}_{\mu\nu}\Sigma^a_{\rho\sigma} + \Sigma^a_{\mu\nu}\chi^{kD}_{\rho\sigma}) \\
& +T^a_{AB}T^b_{CD}\phi^{ABCD}\Sigma^a_{\mu\nu}\Sigma^b_{\rho\sigma}
\end{aligned}
\tag{55}
$$

The Lagrangeans (53) and (54) have two kinds of $U(1)$ "symmetries", one of which is given by

$$\begin{aligned}
&\omega^a_\mu \to \omega^a_\mu, \quad \psi^{kA}_\mu \to \psi^{kA}_\mu, \quad B_\mu \to B_\mu \\
&\Sigma^a_{\mu\nu} \to \mathrm{e}^{\varphi}\Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \to \mathrm{e}^{\varphi}\chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \to \mathrm{e}^{\varphi}\Pi_{\mu\nu} \\
&\phi^{ABCD} \to \mathrm{e}^{-2\varphi}\phi^{ABCD}, \quad \kappa^{ABC} \to \mathrm{e}^{-2\varphi}\kappa^{ABC}, \quad \lambda^{AB} \to \mathrm{e}^{-2\varphi}\lambda^{AB} \\
&g \to \mathrm{e}^{\varphi} g, \quad \Lambda \to \mathrm{e}^{-2\varphi}\Lambda
\end{aligned} \tag{56}$$

We call the another $U(1)$ "symmetry" as $U(1)_R$ "symmetry" since the symmetry corresponding to the scale transformation of the Grassmann number θ^A. The $U(1)_R$ "symmetry" is given by

$$\begin{aligned}
&\omega^a_\mu \to \omega^a_\mu, \quad \psi^{kA}_\mu \to \mathrm{e}^{\rho}\psi^{kA}_\mu, \quad B_\mu \to \mathrm{e}^{2\rho} B_\mu \\
&\Sigma^a_{\mu\nu} \to \Sigma^a_{\mu\nu}, \quad \chi^{kA}_{\mu\nu} \to \mathrm{e}^{-\rho}\chi^{kA}_{\mu\nu}, \quad \Pi_{\mu\nu} \to \mathrm{e}^{-2\rho}\Pi_{\mu\nu} \\
&\phi^{ABCD} \to \phi^{ABCD}, \quad \kappa^{ABC} \to \mathrm{e}^{\rho}\kappa^{ABC}, \quad \lambda^{AB} \to \mathrm{e}^{2\rho}\lambda^{AB} \\
&g \to g, \quad \Lambda \to \left\{ \begin{array}{ll} \mathrm{e}^{2\rho}\Lambda & (N=1) \\ \mathrm{e}^{4\rho}\Lambda & (N=2) \end{array} \right.
\end{aligned} \tag{57}$$

If we assume the above $U(1)$ symmetries survive in the quantum theory, the form of the effective Lagrangean is restricted. If we started from the theory which does not has a cosmological term ($\Lambda = 0$), the gauge symmetry including the left-handed supersymmetry restricts the form of the terms appearing in the effective Lagrangean as $g^l \left(\frac{1}{g}R\right)^m X^n$ after integrating the multiplier field Φ (Here we abbreviated the Lorentz indeces). The $U(1)$ symmetry and Lorentz symmetry give the further restrictions:

$$l - m + n = 0 \tag{58}$$

$$m + n = 2 \tag{59}$$

i.e.,

$$l = 2m - 2 \tag{60}$$

It would be natural to assume the theory has the good weak coupling limit ($g \to 0$), which gives $l \geq 0$. We also assume $m \geq 0$ since R contains the derivative. Then there does not appear the cosmological term even in the quantum theory. The term proportional to R^m appears only perturbatively. Since there does not appear the higher derivative terms perturbatively, the

possible terms are $(l, m, n) = (0, 1, 1)$, $(2, 2, 0)$. Therefore if the term of $(l, m, n) = (2, 2, 0)$ do not appear at the order of g^2, only the term in the original Lagrangean *i.e.*, the term of $(l, m, n) = (0, 1, 1)$ can appear. This might tell only that there is no quantum correction and the Einstein theory is the unique infrared theory.

5 Summary

In this paper, we have considered the group theoretical structure of $N = 1$ and $N = 2$ two-form supergravity theories based on $N = 1$ and $N = 2$ topological superalgebras (TSA), which are the subalgebras of $N = 1$ and $N = 2$ Neveu-Schwarz algebaras whose generators are $(L_0, L_{\pm 1}, G_{\pm\frac{1}{2}})$ and $(L_0, L_{\pm 1}, G^{\pm}_{\pm\frac{1}{2}}, J_0)$, respectively. By introducing two Grassmann variables θ^A $(A = 1, 2)$, we have found the explicit representations of the generators Q^i and we found that any product of the generators is given by a linear combination of the generators; $Q^i Q^j = \sum_k f_k^{ij} Q^k$. By using the expression and the direct product representation, it has been explained how to construct finite-dimensional representation of the gauge groups. It is expected that this gives a clue for the non-perturbative analysis of the supergravity.

Acknowledgement

I would like to appreciate A. Sugamoto for the useful discussions.

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